7494/KA3 **OCTOBER 2008**

Paper III — TOPOLOGY AND MEASURE THEORY

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. If Y is a subspace and X, A is a subset of Y and $\overline{A}$ denote the closure of 'A' in X, prove that the closure of A in Y equals $\overline{A} \cap Y$.

2. Let $f : A \to X \times Y$ be given by the equation, $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if the functions $f_1 : A \to X$ and $f_2 : A \to Y$.

3. If Y is a subspace of X, prove that Y is compact if and only if every covering of Y by sets open in X contains a finite sub collections covering Y.

4. Every metrizable space is normal.

5. If X_0 is a thick subset of a measure space (X, S, μ), if $S_0 = S \cap X_0$ and if, for E in S, $\mu_0(E \cap X_0) = \mu(E)$ then (X_0, S_0, μ_0) is a measure space.

6. If μ^* is an outer measure on a hereditary σ-ring H and of $\overline{S}$ is the class of all μ^*-measurable sets, then $\overline{S}$ is a σ-ring. If $A \in H$ and if $\{E_n\}$ is a disjoint sequence of sets in $\overline{S}$ with $\bigcup_{n=1}^{\infty} E_n = E$, then

$$\mu^*(A \cap E) = \sum_{n=1}^{\infty} \mu^*(A \cap E_n)$$

7. If f is an integrable function such that $\int_F f d\mu = 0$ for every measurable set F, then $f = 0$ a.e.

8. If S and T are rings of subsets of X and Y respectively, then the class R of all finite, disjoint unions of rectangles of the form $A \times B$, where $A \in S$ and $B \in T$ is a ring.

SECTION B — (3 × 20 = 60 marks)

Answer any THREE questions.

9. The product of finitely many compact spaces is compact.

10. Every well-ordered set X is normal in the order topology. – Explain.

11. State and prove that Tietze extension theorem.

12. Let T be the one to one transformation of the entire real line onto itself, defined by $T(x) = \alpha x + \beta$, where α and β are real numbers and $\alpha \neq 0$. If, for every subset E of x, $T(E)$ denotes the set of all points of the form $T(x)$ with x is E, i.e $T(E) = \{\alpha x + \beta : x \in E\}$, then $\mu^*(T(E)) = |\alpha| \mu^*(E)$ and $\mu^*(T(E)) = |\alpha| \mu_*(E)$ The set $T(E)$ is a Borel set or a lebsgue measurable set iff E is a Borel set or a lebsgue measurable set, respectively.

13. State and prove that Radon-Nikodym theorem.

14. If μ is a σ-finite measure on a ring R, then there is a unique measure $\overline{\mu}$ on the σ-ring $S(R)$ such that, for E in R, $\overline{\mu}(E) = \mu(E)$, the measure $\overline{\mu}$ is σ-finite.